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FLUXES, BRANE CHARGES AND CHERN MORPHISMS OF HYPERBOLIC GEOMETRY

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Abstract

The purpose of this paper is to provide the reader with a collection of results which can be found in the mathematical literature and to apply them to hyperbolic spaces that may have a role in physical theories. Specifically we apply K-theory methods for the calculation of brane charges and RR-fields on hyperbolic spaces (and orbifolds thereof). It is known that by tensoring K-groups with the rationals, K-theory can be mapped to rational cohomology by means of the Chern character isomorphisms. The Chern character allows one to relate the analytic Dirac index with a topological index, which can be expressed in terms of cohomological characteristic classes. We obtain explicit formulas for Chern character, spectral invariants, and the index of a twisted Dirac operator associated with real hyperbolic spaces. Some notes for a bivariant version of topological K-theory (KK-theory) with its connection to the index of the twisted Dirac operator and twisted cohomology of hyperbolic spaces are given. Finally we concentrate on lower K-groups useful for description of torsion charges.

Keywords: Fluxes; branes; methods of spectral and K-theory

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1 Introduction

K-theory methods have been applied to classify D-brane charges and RR-fields [1, 2, 3], whereby the latter have been identified with elements of the Grothendieck K-groups [4, 5, 6]. In this paper we are interested on D-branes and fluxes on hyperbolic spaces and orbifolds (see below). Let $X = G/\mathcal{K}$ be an irreducible rank one symmetric space of non-compact type. Thus G will be a connected non-compact simple split rank one Lie group with finite center, and $\mathcal{K} \subset G$ will be a maximal compact subgroup. More concretely the object of interest is the groups $G = SO_1(N, 1)$ $(N \in \mathbb{Z}_+)$ and $\mathcal{K} = SO(N)$. The corresponding symmetric space of non-compact type is the real hyperbolic space $X = \mathbb{H}^N = SO_1(N, 1)/SO(N)$ of sectional curvature -1. The relevant K-theory for the latter is the equivariant one as was shown in [1, 7, 8, 9]. Before going to the specific problem considered in the paper we would like to recall some example of string compactification with non-spherical horizons. De Sitter, anti-de Sitter spaces and $N-$spheres $\mathbb{S}^N$ frequently appear as vacuum solutions in string theory. These spaces (as well as $N-$dimensional real hyperbolic spaces $\mathbb{H}^N$) naturally arise as the near-horizon of black brane geometries. Spheres and the anti-de Sitter spaces, as supergravity solutions, have been extensively investigated. Research on de Sitter solutions has been limited by the fact that they break supersymmetry and, moreover, it is hard to define a quantum field theory on them.

Hyperbolic spaces have attracted much less attention. As an example, let us consider a solution to the equations of motion in eleven-dimensional supergravity which is provided by the Freund-Rubin ansatz for the antisymmetric field strength. The requirement of unbroken supersymmetry, i.e. of vanishing gravitino transformation, is equivalent to the existence of $SO(1, 6)$ and $SO(4)$ Killing spinors θ and η, respectively. The two-form equations admit solutions of the type $X^7 \times Y^4$, where X^7 and Y^4 are Einstein spaces of negative and positive curvature, respectively. But only those spaces that admit Killing spinors preserve supersymmetry. The integrability conditions for spinor equations are $W_{\mu\nu\rho\sigma}\gamma^{\rho\sigma}\theta = 0$, $W_{mnpq}\gamma^{pq}\eta = 0$, where $W_{\mu\nu\rho\sigma}$ and W_{mnpq} are the Weyl tensors of X^7 and Y^4, respectively. Well-known supersymmetric examples for Y^4 include the round four-sphere $\mathbb{S}^4$ and its orbifolds $\Gamma\backslash\mathbb{S}^4$, where Γ is an appropriate discrete group [10]. For the X^7 space one can take the anti de Sitter space AdS_7, which preserves supersymmetry as well, and leads to the $AdS_7\times\mathbb{S}^4$ vacuum of eleven-dimensional supergravity. There are however also solutions involving hyperbolic spaces which are vacua of eleven-dimensional supergravity: $AdS_{7-N}\times\mathbb{H}^N\times\mathbb{S}^4$ $(N \geq 2)$, $AdS_3\times\mathbb{H}^2\times\mathbb{H}^2\times\mathbb{S}^4$, $AdS_2\times\mathbb{H}^2\times\mathbb{H}^3\times\mathbb{S}^4$. Hyperbolic spaces have infinite volume with respect to the Poincaré metric. Thus there are no normalizable modes for any field configurations in hyperbolic spaces. On the other hand non-empty bulk and boundary field theories can exist in coset spaces with topology $\Gamma\backslash\mathbb{H}^N$.

The hyperbolic manifolds $\mathbb{H}^N$, as factors in supergravity solutions, admit Killing spinors. However in the space-time solutions $AdS_3 \times \mathbb{H}^2 \times \mathbb{H}^2 \times \mathbb{S}^4$, $AdS_2 \times \mathbb{H}^2 \times \mathbb{H}^3 \times \mathbb{S}^4$, the factors $\mathbb{H}^2 \times \mathbb{H}^2$, $\mathbb{H}^2 \times \mathbb{H}^3$ cannot leave any unbroken supersymmetry [11]. Although non-supersymmetric, these spaces are an interesting setup for string compactifications, in particular for the construction of conformal field theories. Beside M-theory solutions, one can also have classical type II superstring theory solutions that include hyperbolic factors. These are, for instance, $AdS_3 \times \mathbb{H}^2 \times \mathbb{S}^5$ and $AdS_2 \times \mathbb{H}^3 \times \mathbb{S}^5$. Milne universe type solutions are also found

in type II: they involve hyperbolic spaces of generic dimension [12]. These spaces do not have Killing spinors, therefore they do not preserve any supersymmetry. On a general footing, supersymmetry guarantees stability of the vacuum, but its absence does not necessarily imply instability. In particular D-branes may well represent stable configurations even though they are not BPS solutions. Therefore, in the sequel, we will study the problems of D-branes in hyperbolic spaces and relative orbifolds, abstracting from supersymmetry considerations.

The purpose of this paper is to provide the reader with a collection of results which can be found in the mathematical literature and to apply them to hyperbolic spaces that may have a role in physical theories. We refer in particular to results obtained by cohomological and K-theory methods, which are becoming more and more important in view of the growing demand coming from gauge and gravity and especially string and brane theories.

The paper is organized as follows. In Section 2 we give a simple introduction to topological K-theory. More formal definitions and properties can be found in Appendix, which follows [13] and is designed to provide the reader with the background of topological K-theory, especially in regards to real hyperbolic manifolds. Section 3 is devoted to spectral functions, eta and Chern-Simons invariants of real hyperbolic spaces. K-homology classes describe branes: they can be associated with maps of spin-manifolds X into the spacetime background. The holonomy of RR fields over a brane can be measured by the spectral eta invariant of a vector bundle restricted to X. We consider spectral functions of hyperbolic geometry in Section 3.1. Using the standard definition for the Dirac bundle we review results on the spectral Selberg type zeta functions (or Shintani functions) and the twisted holomorphic eta function of Atiyah-Patodi-Singer. In Section 3.2 we describe the explicit formula for $U(n)-$Chern-Simons invariant of hyperbolic three-spaces. If the three-manifold is a homology three-sphere (like in our case), every $U(n)-$representation of the fundamental group is a $SU(n)-$representation. A connection between the index theory of Dirac operators and topological K-theory is presented in Section 4. Dirac operators are examples of pseudo-differential elliptic operators, which are Fredholm when viewed as operators on a Hilbert space. The indexes of the twisted Dirac operators acting on spaces with topology $X = SO_1(2n,1)/SO(2n)$ are given in Section 4.1. The Kasparov's KK-pairings and the concept of K-amenable groups are considered in Section 5.1. In Section 5.2 we give some notes on K-groups of symmetric spaces, which are useful for the description of D-brane torsion charges. Finally in Section 6 we concentrate on the lower algebraic K-groups of a real compact oriented hyperbolic three-manifold.

2 Aspects of topological K-theory

In this section we would like to review some general ideas and collect some formulas of K-theory in type IIA and type IIB superstring theories, with a view to applying them to a hyperbolic background. Following [13] we have collected a few very basic definitions and properties concerning these groups in Appendix. Here we mostly stick to a rather informal presentation.

It is well-known that type IIB brane charges in a space-time manifold X are well described as elements of $K(X)$ and type IIA brane charges as elements of $K^{-1}(X)$, while RR-fluxes are classified by $K^{-1}(X)$ and $K(X)$, respectively. Let us start from the simplest example: a type

IIB theory in which X is non-compact and is the product of the brane world-volume $\mathbb{R}^{p+1}$ and the transverse space $\mathbb{H}^N$. In this situation the charge is determined by the transverse space[3].

The charge in the simplest cases is just an integral over a form density. At times the form class itself is referred to as the charge and we will also stick to this small abuse of language. In general however cohomology is not enough, [14, 1, 15]: cohomology has to be lifted to K-theory in order to describe brane charges appropriately. A cohomology class is replaced by a K-theory class. So charges are represented by K-theory classes. In the simple example we are considering the relevant space is the transverse space $\mathbb{H}^N$. It is homeomorphic to $\mathbb{R}^N$, therefore its K-theory is going to be the same as for the latter. If $\mathbb{H}^N$ is the transverse space (fibre) to a (non-compact) brane, by definition we set $K(\mathbb{H}^N) \equiv \widetilde{K}(\mathbb{S}^N)$ where $\widetilde{K}$ is the reduced K-theory. Since $\widetilde{K}(\mathbb{S}^N) = \delta_{N,0}\,\mathbb{Z}$ for $N \bmod 2$, we get for $\ell \in \mathbb{Z}_+$ the following result

$$K(\mathbb{H}^N) = \begin{cases} \mathbb{Z}, & N = 2\ell\,, \\ 0, & N = 2\ell + 1\,. \end{cases} \tag{1}$$

Reduced K-groups are formally defined in Appendix: $K(X)$ being the Grothendieck group defined by copies (E, F) of vector bundles over X, the reduced K-group is defined by copies of vector bundles of the same rank. This is physically motivated by the fact that, in the absence of D-branes in the vacuum, we require any two bundles E and F to be isomorphic 'at infinity', [1]. In turn $\tilde{K}(X)$ is isomorphic to $K_c(X)$, the K-group with compact support.

For completeness we write down the only other distinct K-groups (due to periodicity, see Appendix)

$$K^1(\mathbb{H}^N) \equiv \widetilde{K}^1(\mathbb{S}^N) = K^1(\mathbb{S}^N) = K(\mathbb{S}^{N+1}) = \mathbb{Z} \oplus \widetilde{K}(\mathbb{S}^{N+1})\,. \tag{2}$$

These groups are relevant to the classification of RR-fluxes.

As pointed out in the introduction, supergravity solutions which are noncompact in the internal directions are not very appealing. Therefore the previous examples have only a pedagogical motivation. In the sequel we wish to discuss more interesting cases involving compact hyperbolic spaces obtained from non-compact ones by acting some discrete group Γ. In this case the simplified treatment given above is not adequate. First of all the relevant K-theory is the equivariant one, $K(X_\Gamma)$ (see Appendix); moreover the 'transverse directions' must be replaced by the consideration of the normal bundle to the brane world-volume. In general the K-groups have a free part and a torsion part. By tensoring the K-groups with the rationals it is possible to map the K-theory to rational cohomology and obtain very interesting formulas. In this passage a crucial role is played by the Chern character isomorphism. Let us review it.

The Chern character. One of the features of the topological K-theory which makes it so useful in a variety of applications is the existence of the Chern character isomorphism. To set the necessary background for our discussion, we shall briefly review the construction of the

[3]This comes from the equation (Bianchi identity) $dF_p = \delta(W)$ characteristic of a $p-$brane with world-volume W: $\delta(W)$ is $(p+1)-$delta-function-form which is the Poincaré dual of the submanifold W. This form is 'transverse' to W and determines a cohomology class, which gets lifted to a K-theory class in the K-theoretic framework. By transverse directions we refer to the fibres of the normal bundle of W in X, the relevant K-theory being the one with compact support along the fibre.

Chern character in the topological K-theory. Let $\mathbb{E}$ be a complex vector bundle over a compact *topological* space X. There are special cohomology classes of $\mathbb{E}$, the Chern characteristic classes: $c_j(\mathbb{E}) \in H^{2j}(X, \mathbb{Z})$. Their basic properties are the following ones:

(i) $c_0(\mathbb{E}) = 1 \in H^0(X, \mathbb{Z})$;
(ii) For all $n \geq 0, c_n(\mathbb{E} \oplus \mathbb{F}) = \sum_{i+j=n} c_i(\mathbb{E}) \cup c_j(\mathbb{F})$;
(iii) If $f : Y \to X$ is a continuous map, then $c_n(f^*\mathbb{E}) = f^*c_n(\mathbb{E})$.

In fact the axiom (**ii**) is the Whitney sum formula. It implies that the total Chern class $c(\mathbb{E}) = \sum_{j>0} c_j(\mathbb{E}) \in H^{\#}(X, \mathbb{Z})$ depends only on the class of the $\mathbb{E}$ bundle in $K^0(X)$. Therefore c_j extend to functions $c_j : K^0(X) \to H^{2j}(X, \mathbb{Z})$. We can thus, for any complex vector bundle $\mathbb{E}$, define the Chern character ch with the properties:

(iv) $\mathrm{ch}_0(\mathbb{E})$ equals the rank $(\mathbb{E})$ of $\mathbb{E}$, where $\mathrm{rank}(\mathbb{E}) \in H^0(X, \mathbb{Z}) \simeq \mathbb{Z}$;
(v) $\mathrm{ch}(\mathbb{E} \oplus \mathbb{F}) = \mathrm{ch}(\mathbb{E}) + \mathrm{ch}(\mathbb{F})$;
(vi) $\mathrm{ch}(f^*(\mathbb{E})) = f^*(\mathrm{ch}(\mathbb{E}))$ for a continuous map $f : Y \to X$.

The Chern character can be viewed as a natural transformation of functors $K^{\#}(\bullet) \to H^{\#}(\bullet, \mathbb{Q})$. Eliminating torsion by using rational K-theory, we get the following result. Let X be a finite $CW-$complex, then the Chern character $\mathrm{ch} : K^{\#}(X) \otimes \mathbb{Q} \to H^{\#}(X; \mathbb{Q})$ is a natural isomorphism between rational K-theory and rational cohomology. The Chern characters $\mathrm{ch}^{\#}$ (cohomology) and $\mathrm{ch}_{\#}$ (homology) preserve the "cap" product $\cap$. It means that for every *topological space* X there is a $\mathbb{Z}_2-$degree preserving commutative sequence [16, 17]:

$$
\begin{aligned}
&K^{\#}(X) \otimes K_{\#}(X) \xrightarrow{\cap} K_{\#}(X) \xrightarrow{\mathrm{ch}_{\#}} H_{\#}(X, \mathbb{Q}) \\
&H_{\#}(X, \mathbb{Q}) \xleftarrow{\cap} H^{\#}(X, \mathbb{Q}) \otimes H_{\#}(X, \mathbb{Q}) \xleftarrow{\mathrm{ch}^{\#} \otimes \mathrm{ch}_{\#}} K^{\#}(X) \otimes K_{\#}(X)
\end{aligned}
\tag{3}
$$

The Chern isomorphism allows us to arrive at a very nice formula for rational D-brane charges [18, 14]. To this end we need two more tools, the Gysin homomorphism and the Thom isomorphism.

Gysin and Thom maps. Before discussing the brane charge formula, we begin with some conventions which apply throughout. Let X be an oriented manifold, and let $H^{\#}(X)$ be the cohomology ring of X. The Poincaré duality (a well-known result in differential topology) gives a canonical isomorphism

$$
\mathfrak{d}_X : \; H^{\ell}(X) \xrightarrow{\approx} H_{n-\ell}(X), \quad \text{for all} \;\; \ell = 0, 1, \ldots, \;\; n = \dim X \, . \tag{4}
$$

Let $f : Y \to X$ be a continuous map from Y to X and $m = \dim Y$. For all $\ell \geq m - n$ there is a linear map, called the Gysin homomorphism: $f_! : H^{\ell}(Y) \longrightarrow H^{\ell-(m-n)}(X)$, which is defined such that the diagram

$$
\begin{array}{ccc}
H^{\ell}(Y) & \xrightarrow{\mathfrak{d}_Y} & H_{m-\ell}(Y) \\
f_! \downarrow & & \downarrow f_* \\
H^{\ell-(m-n)}(X) & \xleftarrow{\mathfrak{d}_X^{-1}} & H_{m-\ell}(X)
\end{array}
\tag{5}
$$

is commutative. Thus, $f_! = \mathfrak{d}_X^{-1}\, f_*\, \mathfrak{d}_Y$, where f_* is the natural push-forward map acting on homology. As an example of that construction, let us assume that Y is an oriented vector bundle E over X, of fiber dimension ℓ. The canonical projection map, $\pi : E \to X$, and the inclusion $\iota : X \to E$ of the zero section induce maps on the homology with $\pi_* \iota_* = \mathrm{Id}$. For all j, we have the following isomorphisms:

$$\pi_! :\ H^{j+\ell}(E) \xrightarrow{\approx} H^j(X)\,, \quad \iota_! :\ H^j(X) \xrightarrow{\approx} H^{j+\ell}(E)\,.$$

$\pi_!$ is the Gysin map; it can be associated with integration over the fibers of $E \to X$. We have $\pi_! \iota_! = \mathrm{Id}$, so that $\pi_! = (\iota_!)^{-1}$. The map $i_!$ is called the Thom isomorphism of the oriented vector bundle E. The particular example $j = 0$ is an important case of the Thom isomorphism. For $j = 0$, a map $H^0(X) \to H^\ell(E)$ and the image of $1 \in H^0(X)$ determine a cohomology class $\iota_!(1) \in H^\ell(E)$, which is called the Thom class of E.

After this preliminary review, let us consider a $U(n)$ gauge bundle $\mathbb{E}$ on the brane. It has been shown, [14], that, using the Gysin and Thom maps, one can find an explicit expression for rational brane charges. As an element of $H^*(X)$, the RR charge (class) associated with a D-brane wrapping a supersymmetric cycle in spacetime $f : Y \hookrightarrow X$ with Chan-Paton bundle $\mathbb{E} \to Y$, is given by

$$Q = \mathrm{ch}(f_! \mathbb{E}) \wedge [\widehat{A}(TX)]^{1/2}\,. \tag{6}$$

One can obtain charges by integrating the RHS over the appropriate cycles. As explained above the map

$$\mathrm{ch} :\ K(X) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow H^{\mathrm{even}}(X, \mathbb{Q}) \equiv \bigoplus_{n \geq 0} H^{2n}(X, \mathbb{Q}) \tag{7}$$

is an isomorphism, and it can be extended to a ring isomorphism [19], $\mathrm{ch} :\ K^*(X) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\approx} H^*(X, \mathbb{Q})$, which maps $K^{-1}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ onto $H^{\mathrm{odd}}(X, \mathbb{Q})$. Note that the rational cohomology ring $H^{\mathrm{even}}(X, \mathbb{Q})$ has a natural inner product, while the pairing $K(X)$, associated with the cohomology ring $K(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, is given by the index of the Dirac operator. The result (6) is in complete agreement with the fact that the D-brane charge is given by $f_![\mathbb{E}] \in K(X)$, and it gives an explicit formula for the brane charges in terms of the Chern character isomorphism on K-theory.

So far we have been dealing with rational K-theory. The torsion part is more difficult to deal with. However there are instances in which concrete results can be obtained. In the next section we will be exploiting the following remark.

For a finite-dimensional *smooth manifold* X the RR-phase admits a description in terms of the exact sequence [20]:

$$0 \to H^{\mathrm{odd}}(X, \mathbb{R})/[K^1(X)/\mathrm{Tor}] \to K^1(X, U(1)) \to \mathrm{Tor}K^0(X) \to 0$$

The component group is given by the torsion classes in $K^0(X)$, while the trivial component consists of the torus $H^{\mathrm{odd}}(X, \mathbb{R})/(K^1(X)/\mathrm{Tor})$. If $H^{\mathrm{odd}}(X, \mathbb{R})$ vanishes, the group of fluxes is simply $K^1(X, U(1)) \cong \mathrm{Tor}K^0(X)$, and the corresponding RR- fluxes can be represented by flat vector bundles. Since $K^0(X)$ measures charges in type IIB string, this amounts to saying that the flat RR-fluxes $K^1(X, U(1))$ coincide with the torsion charges $\mathrm{Tor}K^0(X)$.

3 Chern classes and Chern-Simons invariants

In the string theory path integral a torsion RR-flux prduces an additional phase factor to a D-brane. A brane can be represented by a K-homology class, that is a map of an additional Chan-Paton vector bundle into the spacetime background [20]. The holonomy of the RR-fields over this brane can be associated with the spectral eta invariant of a vector bundle restricted to X. For X a closed oriented hyperbolic manifold and χ an orthogonal representation of $\pi_1 X$, we suppose that χ is acyclic (it means that the vector space of twisted cohomology classes vanishes). Then the eta invariant can be computed from the twisted cochain complex (see for more detail the next section). In the sequel we consider the spectral functions and explicitly calculate the eta and Chern-Simons invariants of hyperbolic spaces, related to the holonomy of RR-fields over branes.

3.1 Complex spectral functions of hyperbolic geometry

Let $X_\Gamma = \Gamma\backslash G/\mathcal{K}$ be a real compact hyperbolic manifold. The fundamental group of X_Γ acts by covering transformations on X and gives rise to a discrete, co-compact subgroup $\Gamma \subset G$ (Γ is called co-compact if it contains only hyperbolic elements). Let $\mathfrak{D}$ denote a generalized Dirac operator associated to a locally homogeneous Clifford bundle over a compact oriented odd dimensional locally symmetric space X_Γ, whose simply connected cover $\widetilde{X}$ is a symmetric space of noncompact type. The fixed point set of the geodesic flow, acting on the unit sphere bundle $T^1 X_\Gamma$, is a disjoint union of submanifolds X_γ. These submanifolds are parametrized by the nontrivial conjugacy classes $[\gamma] \neq 1$ in $\Gamma = \pi_1 X$. By $\mathcal{E}_1(\Gamma)$ we denote the set of those conjugacy classes $[\gamma]$ for which X_γ has the property that the Euclidean de Rham factor of $\widetilde{X}_\gamma$ is one-dimensional; it means that for $[\gamma] \in \mathcal{E}_1(\Gamma)$, $\widetilde{X}_\gamma \cong \mathbb{R} \times \widetilde{X}'_\gamma$ and the lines $\mathbb{R} \times \{x'\}$, $x' \in \widetilde{X}'_\gamma$ are the axes of γ. Projected down to X_γ, they become closed geodesic c_γ, which foliate X_γ. The 'center' bundle $C\widehat{X}_\gamma$ over the space of leaves $\widehat{X}_\gamma$, which in fact turn out to be an orbifold, is determined by the eigenvalues of absolute value 1 of the linear Poincaré map $P(\gamma)$ [4]. The parallel translation around closed geodesics c_γ gives rise to an orthogonal transformation $\widehat{\tau}_\gamma$ of $C\widehat{X}_\gamma$; $T\widehat{X}_\gamma \subset C\widehat{X}_\gamma$ and we let $N\widehat{X}_\gamma$ denote the orthogonal component of $T\widehat{X}_\gamma$ in $C\widehat{X}_\gamma$. The tangent bundle $T\widehat{X}_\gamma$ corresponds to the eigenvalue 1 of $\widehat{\tau}_\gamma$; $N\widehat{X}_\gamma$ can be decomposed according to the other eigenvalues -1, $\exp(\pm\sqrt{-1}\theta)$ $(0 < \theta < \pi)$. For more details we refer the reader to the paper [21].

The restriction to X_γ of the exterior bundle $\Lambda T_{\mathbb{C}} X_\Gamma$ can be pushed down to a vector bundle $\widehat{\Lambda}_\gamma$ over $\widehat{X}_\gamma$ which splits into a subbundle $\widehat{\Lambda}_\gamma^\pm$ corresponding to the eigenvalue $\pm\sqrt{-1}$ of the symbol $\mathfrak{D}$. The Clifford multiplication induces a homomorphism $\widehat{\sigma}_\gamma^{\mathfrak{D}}$ of vector bundles associated with projection map of $\widehat{X}_\gamma$. We obtain a $\widehat{\tau}_\gamma-$equivariant complex $\widehat{\sigma}_\gamma^{\mathfrak{D}} : \widehat{\Lambda}_\gamma^+ \rightarrow \widehat{\Lambda}_\gamma^-$ over $T\widehat{X}_\gamma$ and a class $[\widehat{\sigma}_\gamma^{\mathfrak{D}}] \in K^0_{\widehat{\tau}_\gamma}(T\widehat{X}_\gamma)$, the $\widehat{\tau}_\gamma-$equivariant K-theory group of $T\widehat{X}_\gamma$. The cohomology class

[4] At each point of X_γ one can consider the restriction $P_h(\gamma)$ of the linear Poincaré map $P(\gamma) = d\Phi_1$, i.e. the differential $d\Phi_{t=1}$ of the geodesic flow Φ_t at $(c_\gamma, \dot{c}_\gamma) \in TX_\gamma$, to the direction normal to the Φ_t.

can be formed as in [4] (Section 3),

$$\mathrm{ch}(\widehat{\sigma}_\gamma^{\mathfrak{D}}(\widehat{\tau}_\gamma)) \in H^{even}(T\widehat{X}_\gamma;\mathbb{C}). \tag{8}$$

Theorem 1 *(H. Moscovici and R. J. Stanton [21]) The following Selberg type function $Z(s,\mathfrak{D})$ can be defined, initially for $\Re(s^2) \gg 0$, by the formula*

$$\log Z(s,\mathfrak{D}) \stackrel{def}{=} \sum_{[\gamma]\in\mathcal{E}_1(\Gamma)} (-1)^q \frac{L(\gamma,\mathfrak{D})}{|\det(I-P_h(\gamma))|^{1/2}} \frac{e^{-s\ell(\gamma)}}{m(\gamma)}, \tag{9}$$

where $q=(1/2)\dim(N\widehat{X}_\gamma)$ is an integer independent of γ, $[\gamma]$ runs over the nontrivial conjugacy classes in Γ, $\ell(\gamma)$ is the length of the closed geodesic c_γ in the free homotopy class corresponding to $[\gamma]$, $m(\gamma)$ is the multiplicity of c_γ. The Lefschetz number $L(\gamma,\mathfrak{D})$ is given by (see, for example, [22] or [21], Eq. (5.5)):

$$L(\gamma,\mathfrak{D}) = \left\{ \frac{\mathrm{ch}(\widehat{\sigma}_\gamma^{\mathfrak{D}}(\widehat{\tau}_\gamma))\mathfrak{R}(N\widehat{X}_\gamma(-1)) \prod_{0<\theta<\pi} \mathfrak{S}^\theta(N\widehat{X}_\gamma(\theta))\mathfrak{T}(\widehat{X}_\gamma)}{\det(I-\widehat{\tau}_\gamma|N\widehat{X}_\gamma)} \right\} [T\widehat{X}_\gamma]. \tag{10}$$

The Lefschetz formula (10) has been given using the stable characteristic classes $\mathfrak{R}, \mathfrak{S}^\theta$ and $\mathfrak{T}$ defined in [23] (Theorem 3.9).

Furthermore $\log Z(s,\mathfrak{D})$ has a meromorphic continuation to $\mathbb{C}$ given by the identity

$$\log Z(s,\mathfrak{D}) = \mathrm{logdet}'\left(\frac{\mathfrak{D}-\sqrt{-1}s}{\mathfrak{D}+\sqrt{-1}s}\right) + \sqrt{-1}\pi\eta(s,\mathfrak{D}), \tag{11}$$

where $s\in\sqrt{-1}\mathrm{Spec}'(\mathfrak{D})$ $(\mathrm{Spec}'\equiv\mathrm{Spec}(\mathfrak{D})-\{0\})$, and $Z(s,\mathfrak{D})$ satisfies the functional equation

$$Z(s,\mathfrak{D})Z(-s,\mathfrak{D}) = e^{2\pi\sqrt{-1}\eta(s,\mathfrak{D})}. \tag{12}$$

Generalized Dirac operator. From now on we restrict ourselves to bundles that satisfy a local homogeneity condition: a vector bundle E over X_Γ is $\mathcal{G}-$locally homogeneous, for some Lie group $\mathcal{G}$, if there is a smooth action of $\mathcal{G}$ on E which is linear on the fibers and covers the action of $\mathcal{G}$ on $\widetilde{X}$. Standard constructions from linear algebra applied to any $\mathcal{G}-$locally homogeneous E give in a natural way corresponding $\mathcal{G}-$locally homogeneous vector bundles. In particular, all bundles $TX_\Gamma, \mathbb{C}\ell(X_\Gamma), \mathrm{End}\,E\simeq E^*\otimes E$ are $\mathcal{G}-$locally homogeneous [21]. We suppose that all constructions associated with $\mathcal{G}-$locally homogeneous bundles are $\mathcal{G}-$equivariant. Let $\mathfrak{D}$ denote a generalized Dirac operator associated to a locally homogeneous bundle E over X_Γ. We shall require $\mathcal{G}-$equivariance for the natural connection ∇ so that the corresponding Dirac operator is $\mathcal{G}-$invariant.

Suppose now that $\chi:\Gamma\to U(F)$ be a unitary representation of Γ in F. The Hermitian vector bundle $\mathbb{F}=X\times_\Gamma F$ over X_Γ inherits a flat connection from the trivial connection on $X\times F$. If

$\mathfrak{D} : C^\infty(X,V) \to C^\infty(X,V)$ is a differential operator acting on the sections of the vector bundle V, then $\mathfrak{D}$ extends canonically to a differential operator $\mathfrak{D}_\chi : C^\infty(X, V\otimes\mathbb{F}) \to C^\infty(X, V\otimes\mathbb{F})$, uniquely characterized by the property that $\mathfrak{D}_\chi$ is locally isomorphic to $\mathfrak{D}\otimes ...\otimes\mathfrak{D}$ (dim F times). We specialize to the case of locally homogeneous Dirac operators $\mathfrak{D} : C^\infty(X_\Gamma, \mathbb{E}) \to C^\infty(X_\Gamma, \mathbb{E})$ in order to construct a generalized operator $\mathfrak{D}_\chi$, acting on spinors with coefficients in χ (see for detail [21]). One can repeat the arguments of the previous discussion to construct a twisted zeta function $Z(s, \mathfrak{D}_\chi)$.

Theorem 2 *(H. Moscovici and J. J. Stanton [21]). There exists a zeta function $Z(s, \mathfrak{D}_\chi)$, meromorphic on $\mathbb{C}$, given for $\Re(s^2) \gg 0$ by the formula*

$$\log Z(s, \mathfrak{D}_\chi) \stackrel{def}{=} \sum_{[\gamma]\in\mathcal{E}_1(\Gamma)} (-1)^q \mathrm{Tr}\chi(\gamma) \frac{L(\gamma, \mathfrak{D})}{|\det(I - P_h(\gamma))|^{1/2}} \frac{e^{-s\ell(\gamma)}}{m(\gamma)}; \tag{13}$$

moreover, one has

$$\log Z(0, \mathfrak{D}_\chi) = \sqrt{-1}\pi\eta(0, \mathfrak{D}_\chi). \tag{14}$$

3.2 $U(n)-$gauge bundles and the Chern-Simons invariants

The Chern character allows one to map the analytical Dirac index in terms of K-theory classes into a topological index which can be expressed in terms of cohomological characteristic classes. This results in a connection between the Chern-Simons action and the celebrated Atiyah-Singer index theorem. The goal of this section is to present explicit formulas for the Chern classes and $U(n)-$Chern-Simons invariant of an irreducible flat connection on real compact hyperbolic three-manifolds.

The Chern-Simons functional CS as a function on the space of connections on a trivial principal bundle over a compact oriented three-manifold X_Γ is given by

$$CS(A) = \frac{1}{8\pi^2}\int_{X_\Gamma} \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right).$$

Let $\mathcal{P} = X_\Gamma \otimes \mathfrak{G}$ be a principal bundle over X_Γ with the gauge group $\mathfrak{G} = U(n)$ and let $\mathfrak{A}_{X_\Gamma} = \Omega^1(X_\Gamma; \mathfrak{g})$ be the space of all connections on $\mathcal{P}$; this space is an affine space of one-forms on X_Γ with values in the Lie algebra $\mathfrak{g}$ of $\mathfrak{G}$. The value of the function $CS(A)$ on the space of connections $\mathfrak{A}_{X_\Gamma}$ can be regarded as a topological invariant of a pair (X_Γ, χ), where χ is an orthogonal representation of the fundamental group Γ. Let $U_{X_\Gamma} = \{A \in \mathfrak{A}_{X_\Gamma} | F_A = dA + A\wedge A = 0\}$ be the space of flat connections on $\mathcal{P}$.

A well-known formula related to the CS integrand is: $d\mathrm{Tr}(A\wedge dA + (2/3)A\wedge A\wedge A) = \mathrm{Tr}\left(F_A\wedge F_A\right)$. This formula permits another approach to the Chern-Simons invariant. Indeed, let M be an oriented four-manifold with boundary $\partial M = X_\Gamma$. One can extend $\mathcal{P}$ to a trivial $\mathfrak{G}-$bundle over M; then Stokes' theorem gives

$$CS(\widetilde{A}) = \frac{1}{8\pi^2}\int_M \mathrm{Tr}\left(F_{\widetilde{A}}\wedge F_{\widetilde{A}}\right), \tag{15}$$

where $\widetilde{A}$ is any extension of A over M. This can be viewed as a generalization of the Chern-Simons invariant to the case in which $\mathcal{P}$ is a non-trivial $U(n)-$bundle over X_Γ. Suppose that χ is any one-dimensional representation of a Γ factor through a representation of $H^1(X;\mathbb{Z})$. It can be shown that for a unitary representation $\chi : \Gamma \to U(n)$, the corresponding flat vector bundle $\mathbb{E}_\chi$ is topologically trivial ($\mathbb{E}_\chi \cong X \otimes \mathbb{C}^n$) if and only if

$$\det \chi|_{\mathrm{Tor}^1} : \mathrm{Tor}^1 \to U(1) \tag{16}$$

is the trivial representation. Here Tor^1 is the torsion part of $H^1(M;\mathbb{Z})$ and $\det\chi$ is a one-dimensional representation of Γ defined by $\det\chi(\gamma) := \det(\chi(\gamma))$, for $\gamma \in \Gamma$. If $\widetilde{A}_\chi$ is an extension of a flat connection A_χ corresponding to χ, the second Chern character $\mathrm{ch}_2(\widetilde{\mathbb{E}}_\chi)$ $(= -(1/8\pi^2)\mathrm{Tr}(F_{\widetilde{A}_\chi} \wedge F_{\widetilde{A}_\chi}))$ of $\widetilde{\mathbb{E}}_\chi$ can be expressed in terms of the first and second Chern classes: $\mathrm{ch}_2(\widetilde{\mathbb{E}}_\chi) = (1/2)c_1(\widetilde{\mathbb{E}}_\chi)^2 - c_2(\widetilde{\mathbb{E}}_\chi)$. The Chern character and the $\widehat{A}-$genus, the usual polynomial related to Riemannian curvature Ω^M, $\widehat{A}(\Omega^M) = \sqrt{\det(\frac{\Omega^M/4\pi}{\mathrm{sinh}\Omega^M/4\pi})}$, are given by

$$\begin{aligned}
\mathrm{ch}(\widetilde{\mathbb{E}}_\chi) &= \mathrm{rank}\,\widetilde{\mathbb{E}}_\chi + c_1(\widetilde{\mathbb{E}}_\chi) + \mathrm{ch}_2(\widetilde{\mathbb{E}}_\chi) = \dim\chi + c_1(\widetilde{\mathbb{E}}_\chi) + \mathrm{ch}_2(\widetilde{\mathbb{E}}_\chi), \\
\widehat{A}(\Omega^M) &= 1 - \frac{1}{24}p_1(\Omega^M).
\end{aligned} \tag{17}$$

Here $p_1(\Omega^M)$ is the first Pontrjagin class, Ω^M is the Riemannian curvature of a four-manifold M which has a boundary $\partial M = X_\Gamma$. Thus we have

$$\begin{aligned}
\mathrm{ch}(\widetilde{\mathbb{E}}_\chi)\widehat{A}(\Omega^M) &= (\dim\chi + c_1(\widetilde{\mathbb{E}}_\chi) + \mathrm{ch}_2(\widetilde{\mathbb{E}}_\chi))(1 - \frac{1}{24}p_1(\Omega^M)) \\
&= \dim\chi + c_1(\widetilde{\mathbb{E}}_\chi) + \mathrm{ch}_2(\widetilde{\mathbb{E}}_\chi) - \frac{\dim\chi}{24}p_1(\Omega^M).
\end{aligned} \tag{18}$$

The integral over the manifold M takes the form

$$\int_M \mathrm{ch}(\widetilde{\mathbb{E}}_\chi)\widehat{A}(\Omega^M) = \int_M \mathrm{ch}_2(\widetilde{\mathbb{E}}_\chi) - \frac{\dim\chi}{24}\int_M p_1(\Omega^M). \tag{19}$$

The following result holds:

Theorem 3 *(M. F. Atiyah, V. K. Patodi and I. M. Singer [24, 25, 26]). The Dirac index is given by*

$$\mathrm{Index}\,\mathfrak{D}_{\widetilde{A}_\chi} = \int_M \mathrm{ch}(\widetilde{\mathbb{E}}_\chi)\widehat{A}(M) - \frac{1}{2}(\eta(0,\mathfrak{D}_\chi) + h(0,\mathfrak{D}_\chi)), \tag{20}$$

where $h(0,\mathfrak{D}_\chi)$ is the dimension of the space of harmonic spinors on X_Γ ($h(0,\mathfrak{D}_\chi) = \mathrm{dimKer}\,\mathfrak{D}_\chi$ = multiplicity of the 0-eigenvalue of $\mathfrak{D}_\chi$ acting on X_Γ); $\mathfrak{D}_\chi$ is a Dirac operator on X_Γ acting on spinors with coefficients in χ.

The Chern-Simons action can be derived from Eq. (20),

$$\mathrm{Index}\,\mathfrak{D}_{\widetilde{A}_\chi} = CS_{U(n)}(\widetilde{A}_\chi) - \frac{\dim\chi}{24}\int_M p_1(\Omega^M) - \frac{1}{2}(\eta(0,\mathfrak{D}_\chi) + h(0,\mathfrak{D}_\chi)). \tag{21}$$

For type IIA a three form flux associates a phase to a Euclidean D2-brane world-volume X_Γ which is given by the eta invariant of the virtual bundle $\eta_{E_\chi} - \eta_{E_{\chi'}}$ restricted to X_Γ. We can express this phase directly in terms of the Chern-Simons invariant. Indeed, for a trivial representation χ_0 one can take a trivial flat connection $\widetilde{A}_{\chi_0}$; then $F_{\widetilde{A}_{\chi_0}} = 0$ and for this choice we get $\mathrm{Index}\,\mathfrak{D}_{\widetilde{A}_{\chi_0}} = -(1/24)\int_M p_1(\Omega^M) - (1/2)(\eta(0,\mathfrak{D}) + h(0,\mathfrak{D}))$. Using this formula in (21) one can obtain

$$\mathrm{Index}\,\mathfrak{D}_{\widetilde{A}_\chi} - \mathrm{dim}\chi\,\mathrm{Index}\,\mathfrak{D}_{\widetilde{A}_{\chi_0}} = CS_{U(n)}(\widetilde{A}_\chi) - \frac{1}{2}(\eta(0,\mathfrak{D}_\chi) - \mathrm{dim}\chi\eta(0,\mathfrak{D}))\ \mathrm{modulo}(\mathbb{Z}/2). \quad (22)$$

Finally we get the following result for the invariant of an irreducible flat connection on the real hyperbolic three-manifolds

$$\begin{aligned}\frac{1}{2}\left(\mathrm{dim}\chi\,\eta(0,\mathfrak{D}) - \eta(0,\mathfrak{D}_\chi)\right) &= \frac{1}{2\pi\sqrt{-1}}\log\left[\frac{Z(0,\mathfrak{D})^{\mathrm{dim}\chi}}{Z(0,\mathfrak{D}_\chi)}\right] \\ &= CS_{U(n)}(\widetilde{A}_\chi)\ \mathrm{modulo}(\mathbb{Z}/2), \qquad (23)\end{aligned}$$

since the index of a Dirac operator, acting on some spin four-manifold, is an integer.

4 Dirac operators and K-theory class

In the previous section we have calculated the Chern character, the Chern classes and the Chern invariants of (3d) hyperbolic geometry, using the fact that the index of a Dirac operator, acting on some spin four-manifold, is an integer. Here we would like to give explicit formulas for the index of a Dirac operator (and twisted Dirac operator) on an even dimensional manifold. First we need to explain the intimate connection between the index theory of Fredholm operators and topological K-theory. Let $\mathcal{O}$ be a *Fredholm operator*, acting on a separable Hilbert space $\mathcal{H}$. It is a bounded linear operator whose kernel (and cokernel) are finite dimensional subspaces of $\mathcal{H}$. Such operators have a well-defined *Index*, which is invariant under perturbations by any compact operator $\mathcal{A}$:

$$\begin{aligned}&\mathrm{Index}\,(\mathcal{O}+\mathcal{A}) = \mathrm{Index}\,(\mathcal{O});\\ &\mathrm{Index}\,(\mathcal{O}) = \mathrm{Index}\,\mathcal{A}\ \text{ if }\ \mathcal{A}\ \text{ is sufficiently close in the operator norm to }\ \mathcal{O}.\end{aligned}$$

These formulas are important since one can describe the group $K(X)$ (and groups of cohomology) in terms of Fredholm operators. Indeed, let $\mathfrak{X}$ be the space of Fredholm operators on $\mathcal{H}$ with the operator norm topology. Then the index of a Fredholm operator defines a continuous map $\mathrm{Index}: \mathfrak{X} \rightarrow \mathbb{Z}$ which induces a bijection [13]: $\pi_0(\mathfrak{X}) \rightarrow \mathbb{Z}$ between the set of connected components of $\mathfrak{X}$ and the integers. Let X be a *compact topological space*. Suppose $[X,\mathfrak{X}]$ be a set of homotopy classes of maps from X to $\mathfrak{X}$. The product of two Fredholm operators is again a Fredholm operator, which implies that $[X,\mathfrak{X}]$ is a monoid. There is an isomorphism [13]: $[X,\mathfrak{X}] \stackrel{\approx}{\rightarrow} K(X)$. Let $\{\mathcal{O}\}_{x\in X}$ be a continuous family of Fredholm operators. The family of vector spaces $\mathrm{Ker}\,\mathcal{O}_x$ ($\mathrm{coKer}\,\mathcal{O}_x$) forms a vector bundle $\mathrm{Ker}\,\mathcal{O}$ ($\mathrm{coKer}\,\mathcal{O}$) over X, and we can

define the index of a family of operators $\mathcal{O}_x$ as the class Index $\mathcal{O} \equiv [(\mathrm{Ker}\,\mathcal{O}\,,\, \mathrm{coKer}\,\mathcal{O})] \in K(X)$. Thus, the composition of operators in $\mathcal{O}$ corresponds to the addition in $K(X)$, while adjoint operation corresponds to inversion. In the case where X is a point, $K(X = pt) = \mathbb{Z}$, the isomorphism $[pt, \mathfrak{X}] \stackrel{\approx}{\to} \mathbb{Z}$ is just the index map $\pi_0(\mathfrak{X}) \to \mathbb{Z}$, the virtual dimension of the K-theory class coincides with the index of a Fredholm operator, i.e. : $\mathrm{ch}_0(\mathrm{Index}\,\mathcal{O}) = \mathrm{Index}\,\mathcal{O}$.

We are interested in applying these ideas to a special class of operators, the Dirac operators associated to vector bundles over a spin manifold X (a standard material on Dirac bundles the reader can find for example in the book [27]). Dirac operators are examples of pseudo-differential elliptic operators, which are Fredholm operators when viewed as operators on a Hilbert space.

4.1 The index of Dirac operator

Taking into account our interest in hyperbolic geometry, we compute in this section the index of Dirac operator acting on real closed hyperbolic manifold (which is in connection with Chern-Simons and eta topological invariants, as we have seen in previous section). We actually consider the special case $X = G/\mathcal{K}$, $G = SO_1(2n, 1)$, $\mathcal{K} = SO(2n)$. The complexified Lie algebra $\mathfrak{g} = \mathfrak{g}_0^{\mathbb{C}} = \mathfrak{s}o(2n+1, \mathbb{C})$ of G is of the Cartan type B_n. Let $\mathfrak{a}_0, \mathfrak{n}_0$ denote the Lie algebras of A, N in an Iwasawa decomposition $G = \mathcal{K}AN$. Since the rank of G is one, $\mathrm{dim}\mathfrak{a}_0 = 1$ by definition, say $\mathfrak{a}_0 = \mathbb{R}H_0$ for a suitable basis vector H_0: $H_0 = \mathrm{antidiag}(1, 1, ..., 1)$. By this choice we have the normalization $\beta(H_0) = 1$, where $\beta : \mathfrak{a}_0 \to \mathbb{R}$ is the positive root which defines $\mathfrak{n}_0$. The standard systems of positive roots $\triangle^+, \triangle_\ell^+$ for $\mathfrak{g}$ and $\mathfrak{k} = \mathfrak{k}_0^{\mathbb{C}}-$ the complexified Lie algebra of $\mathcal{K}$, with respect to a Cartan subgroup H of G, $H \subset \mathcal{K}$, are given by $\triangle^+ = \{\varepsilon_i | 1 \leq i \leq n\} \bigcup \triangle_\ell^+$, where $\triangle_\ell^+ = \{\varepsilon_i \pm \varepsilon_j | 1 \leq i < j \leq n\}$, and $\triangle_n^+ \stackrel{def}{=} \{\varepsilon_i | 1 \leq i \leq n\}$ is the set of positive non-compact roots. Here $(\varepsilon_i,\, \varepsilon_j) = \delta_{ij}[(H_0, H_0)]^{-1} = \delta_{ij}[2(2n-1)]^{-1}$, $(\varepsilon_i \pm \varepsilon_j,\ \varepsilon_i \pm \varepsilon_j) = [2n+1]^{-1}$, $i < j$; i.e. $(\alpha, \alpha) = [2n-1]^{-1}$, $\forall \alpha \in \triangle_n^+$.

Let $\mathfrak{h}_0$ be the Lie algebra of H and let $\mathfrak{h}_{\mathbb{R}}^* = \mathrm{Hom}(\sqrt{-1}\mathfrak{h}_0, \mathbb{R})$ be the dual space of the real vector space $\sqrt{-1}\mathfrak{h}_0$. Thus, the $\{\varepsilon_i\}_{i=1}$ are an $\mathbb{R}$-basis of $\mathfrak{h}_{\mathbb{R}}^*$. Of interest are the *integral* elements f of $\mathfrak{h}_{\mathbb{R}}^*$:

$$f \stackrel{def}{=} \{\mu \in \mathfrak{h}_{\mathbb{R}}^* | \frac{2(\mu, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z},\ \ \forall \alpha \in \triangle^+\}. \tag{24}$$

Then we have $2(\mu, \varepsilon_i)[(\varepsilon_i, \varepsilon_i)]^{-1} = 2\mu_i$, for $1 \leq i \leq n$, $2(\mu, \varepsilon_i \pm \varepsilon_j)[(\varepsilon_i \pm \varepsilon_j, \varepsilon_i \pm \varepsilon_j)]^{-1} = \mu_i \pm \mu_j$, for $1 \leq i < j \leq n$, where $\mu = \sum_{j=1}^n \mu_j \varepsilon_j$ for $\mu \in \mathfrak{h}_{\mathbb{R}}^*$, $\mu_j \in \mathbb{R}$. Let $\delta_k = (1/2)\sum_{\alpha \in \triangle_k^+} \alpha$, $\delta_n = (1/2)\sum_{\alpha \in \triangle_n^+} \alpha$, $\delta = \delta_k + \delta_n = (1/2)\sum_{\alpha \in \triangle^+} \alpha$, then $\delta_k = \sum_{i=1}^n (n-i)\varepsilon_i$, $\delta_n = (1/2)\sum_{i=1}^n \varepsilon_i$, $\delta = \sum_{i=1}^n (n - i - 1/2)\varepsilon_i$ are all integral. The elements μ of f correspond to characters e^μ of H. Following [28] we fix, once and for all, a $\mu \in f$, which is $\triangle_k^+$-dominant: $(\mu, \alpha) \geq 0$ for $\alpha \in \triangle_k^+$. For us, in concrete terms, this means the following. Let $\mu = (\mu_1, ..., \mu_n)$ be a sequence of real numbers such that

(1) $2\mu_i \in \mathbb{Z}$ for $1 \leq i \leq n$ and $\mu_i \pm \mu_j \in \mathbb{Z}$ for $1 \leq i < j < n$ (i.e. μ is integral),

(2) $|\mu_n| \leq \mu_{n-1} \leq \mu_{n-2} \leq ... \leq \mu_2 \leq \mu_1$ and either every $\mu_i \in \mathbb{Z}$ or, else,

every μ_i is half integer.

Note that, in fact, (1) $\rightarrow$ (2) so that we may drop the condition (1) (only for $G = SO_1(2n,1)$ since in general (2) does not imply (1)). Then $\mu + \delta_n$ does define a character of H, as has been required in [28] and, by construction, there is a twisted Dirac operator $\mathfrak{D}_{\mu,\Gamma}$ on a vector bundle over $\Gamma\backslash G/\mathcal{K}$ for Γ a discrete subgroup of G. Thus, $\Gamma\backslash G$ does not need to be compact, but one requires of course that $\mathrm{Vol}(\Gamma\backslash G) < \infty$.

Twisted Dirac operator. In summary, we assume the following: $\mu = (\mu_1, ..., \mu_n)$ is a sequence of real numbers, and

(3) $0 \leq \mu_n \leq \mu_{n-1} \leq \mu_{n-2} \leq \cdots \leq \mu_2 \leq \mu_1$, where either every $\mu_j \in \mathbb{Z}$ or every μ_j has the form $\mu_j = n_j + 1/2$ for some $n_j \in \mathbb{Z}$.

Then we have a twisted Dirac operator $\mathfrak{D}_{\mu,\Gamma}$ over $\Gamma\backslash G/\mathcal{K}$.

Theorem 4 *([28], for the case $G = SO_1(2n,1), n \geq 2$). For a suitable normalization of the Haar measure on G, and for μ satisfying the condition (3), one has*

$$\mathrm{Index}\,\mathfrak{D}_{\mu,\Gamma} = \mathrm{Vol}(\Gamma\backslash G)\frac{\prod_{\alpha\in\triangle^+}(\mu+\delta_k,\alpha)}{\prod_{\alpha\in\triangle_k^+}(\delta_k,\alpha)}. \tag{25}$$

To make this explicit, we must express $P_k \stackrel{def}{=} \prod_{\alpha\in\triangle_k^+}(\delta_k,\alpha)$ and $P \stackrel{def}{=} \prod_{\alpha\in\triangle^+}(\mu+\delta,\alpha)$ directly in terms of the real numbers $\mu_1, ..., \mu_n$. First we compute $P_k \stackrel{def}{=} \prod_{\alpha\in\triangle_k^+}(\delta_k,\alpha)$. Note that the number of pairs (i,j) with $1 \leq i < j \leq n$ is $n(n-1)/2$; $(\delta_k, \delta_j) = \sum_{i=1}^n (n-i)(\varepsilon_i, \varepsilon_j) = [n-j][2(2n-1)]^{-1} \Longrightarrow (\delta_k, \varepsilon_i \pm \varepsilon_j) = [n-i \pm (n-j)][2(2n-1)]^{-1}$, or

$$\begin{aligned} P_k &= \prod_{1\leq i<j\leq n}(\delta_k, \varepsilon_i+\varepsilon_j)\prod_{1\leq i<j\leq n}(\delta_k,\varepsilon_i-\varepsilon_j) = \prod_{1\leq i<j\leq n}\frac{2n-i-j}{2(2n-1)}\prod_{1\leq i<j\leq n}\frac{-i+j}{2(2n-1)} \\ &= \left[\frac{1}{2(2n-1)}\right]^{n(n-1)}\prod_{1\leq i<j\leq n}(2n-i-j)(-i+j). \end{aligned} \tag{26}$$

We also look at $P \stackrel{def}{=} \prod_{\alpha\in\triangle^+}(\mu+\delta_k,\alpha)$; $\mu + \delta_k = \sum_{j=1}^n(\mu_j+n-j)\varepsilon_j \quad \Longrightarrow (\mu+\delta_k, \varepsilon_i) = \sum_{j=1}^n(\mu_j+n-j)(\varepsilon_j,\varepsilon_i) = (\mu_i+n-i)[2(2n-1)]^{-1}$. Then, for $i < j$, $(\mu+\delta_k, \varepsilon_i+\varepsilon_j) = (\mu_i+\mu_j+2n-i-j)[2(2n-1)]^{-1}$, $(\mu+\delta_k,\varepsilon_i-\varepsilon_j) = (\mu_i-\mu_j+i-j)[2(2n-1)]^{-1}$, and

$$\begin{aligned} \prod_{\alpha\in\triangle_k^+}(\mu+\delta_k,\alpha) &= \prod_{1\leq i\leq j\leq n}\frac{\mu_i+\mu_j+2n-i-j}{2(2n-1)}\prod_{1\leq i\leq j\leq n}\frac{\mu_i-\mu_j-i+j}{2(2n-1)} \\ &= \left[\frac{1}{2(2n-1)}\right]^{n(n-1)}\prod_{1\leq i\leq j\leq n}(\mu_i+\mu_j+2n-i-j)(\mu_i-\mu_j-i+j). \end{aligned} \tag{27}$$

Thus we have

$$\prod_{\alpha\in\triangle_n^+}(\mu+\delta_k,\alpha)=\prod_{i=1}^{n}\frac{(\mu_i+n-i)}{2(2n-1)}=\left[\frac{1}{2(2n-1)}\right]^n\prod_{i=1}^{n}(\mu_i+n-i), \tag{28}$$

and therefore,

$$P=\left[\frac{1}{2(2n-1)}\right]^{n^2}\prod_{i=1}^{n}(\mu_i+n-i)\prod_{1\leq i<j\leq n}(\mu_i+\mu_j+2n-i-j)(\mu_i-\mu_j-i+j). \tag{29}$$

We arrive at the final result for $\mu=(\mu_1,...,\mu_n)\in\mathbb{R}^n$, subject to the condition (3) (for a suitable Haar measure on G). The final result follows from formulas (25), (28) and (29). The L^2-index of the twisted Dirac operator $\mathfrak{D}_{\mu,\Gamma}$ is equal to

$$\begin{aligned}\mathrm{Index}\,\mathfrak{D}_{\mu,\Gamma} &= \mathrm{Vol}(\Gamma\backslash G)\frac{P}{P_k}=\frac{\mathrm{Vol}(\Gamma\backslash G)}{[2(2n-1)^n]}\prod_{i=1}^{n}(\mu_i+n-i)\\ &\times\frac{\prod_{1\leq i<j\leq n}(\mu_i+\mu_j+2n-i-j)(\mu_i-\mu_j-i+j)}{\prod_{1\leq i<j\leq n}(2n-i-j)(-i+j)}.\end{aligned} \tag{30}$$

5 Brane charges, KK-groups and K-amenable groups

In many instances of field and string theory, torsion can make its appearance in the integral cohomology groups $H^{\#}(X,\mathbb{Z})$. But local fields are represented by differential forms, consequently by themselves they cannot carry torsion. In previous sections, in particular, we saw that the Chern isomorphism determines the free part of the K-groups from ordinary cohomology – a well-known result, which allows us the avoid more sophisticated mathematical concepts. We also saw that in some fortunate circumstances we can express torsion charges by means of the η invariant. However in general, for suitable spaces X (for example, topological manifolds or finite CW- complexes) $K(X)$ is a finitely generated abelian group, it has the form $\mathbb{Z}^\ell\oplus\mathrm{Tor}$, and in order to fully determine RR–fields and charges we have to resort to full-fledged K-theory.

In this section we would like to broach this subject as far as K-theory on hyperbolic spaces is concerned. This requires the introduction of new mathematical tools: K-homology, KK-theory and K-amenabilty, which will allow us to state some results on twisted K-theory on hyperbolic spaces.

5.1 KK-groups

Given a manifold X let $C(X)$ be the commutative C^*-algebra [5] of all continuous complex-valued functions which vanish at infinity on X. The C^*-algebra, which categorically encodes the topological properties of manifold X, plays a dual role to X in the K-theory of X by the

[5] Recall that a C^*-algebra is a Banach algebra with an involution satisfying the relation $||a^*||=||a||^2$.

Serre-Swan theorem [29]: $K^\ell(X) \cong K_\ell(C(X)), \ell = 0,1$. Here $K^\ell(X)$ is the reduced topological K-theory of X (see section Appendix for details). Let X have a $Spin^{\mathbb{C}}-$structure, then there is a Poincaré duality isomorphism [30]: $K^{\dim X-\ell}(X) \cong K_\ell^{\mathbb{C}}(X)$, $\ell = 0,1$, where $K_\ell^{\mathbb{C}}$ denotes the dual compactly supported K-homology of X.

For a finite dimensional manifold X the exists another C^*-algebra, which is non-commutative and can be constructed with the help of the Riemannian metric g. In fact, we can form the complex Clifford algebra $\mathrm{Cliff}(T_xX, g_x)$, where for each $x \in X$ the tangent space T_xX of X is a finite-dimensional Euclidean space with inner product g_x. This algebra has a canonical structure as a finite-dimensional $\mathbb{Z}_2-$graded C^*-algebra. Let us consider the family of C^*-algebras $\{\mathrm{Cliff}(T_xX, g_x\}_{x\in X}$; it forms a $\mathbb{Z}_2-$graded C^*-algebra vector bundle $\mathrm{Cliff}(TX) \to X$, called the Clifford algebra bundle of X [31]. Let us define $\mathcal{C}(X) = C(X, \mathrm{Cliff}(TX))$ to be the C^*-algebra of continuous sections of the Clifford algebra bundle of X vanishing at infinity. If the manifold X is even-dimensional and has a $Spin^{\mathbb{C}}-$structure then this C^*-algebra is Morita equivalent to $C(X)$. However, in general, $\mathcal{C}(X)$ is Morita equivalent to $C(TX)$. Because of the Morita equivalence of K-theory, it follows that $K_\ell(\mathcal{C}(X)) \cong K_\ell(C(X)) \cong K^\ell(X), \ell = 0,1$. But for odd-dimensional and spin manifold X this relation is more complicated.

KK-pairing. Let us recall that the definition of K-homology involves classifying extensions of the algebra of continuous functions $C(X)$ on the manifold X by the algebra of compact operators up to unitary equivalence [32]. The set of homotopy classes of operators defines the K-homology group $K_0(X)$, and the duality with K-theory is provided by the natural bilinear pairing $([E]\,,\,[\mathfrak{D}]) \mapsto \mathrm{Index}\,\mathfrak{D}_E \in \mathbb{Z}\,,$ where $[E] \in K(X)$ and $\mathfrak{D}_E$ denotes the action of the Fredholm operator $\mathfrak{D}$ on the Hilbert space $\mathcal{H} = L^2(U(X,E))$ of square-integrable sections of the vector bundle $E \to X$ as $\mathfrak{D} : U(X,E) \to U(X,E)$. Hence, it assumes that the KK-pairing may be the most natural framework for our purpose. The KK-theory is a bivariant version of topological K-theory. It provides a useful framework for the study of the index theory.

The main object here is the group $KK(A,B)$, which depends on a pair of graded algebras A and B. Let A and B be C^*-algebras. A pair $(\mathcal{E}, \pi)$, where $\mathcal{E}$ is a $\mathbb{Z}/2\mathbb{Z}$ graded Hilbert $B-$module acted upon by A through a $*-$homomorphism $\pi : A \to \mathcal{L}(\mathcal{E}) = \mathrm{End}^*(\mathcal{E})$, $\forall a \in A$ the operator $\pi(a)$ being of degree $0, \pi(A) \subset \mathcal{L}(\mathcal{E})^{(0)}$, will be called an $(A,B)-$bimodule. Let $E(A,B)$ be a triple $(\mathcal{E}, \pi, F)$, where $(\mathcal{E}, \pi)$ is a $A, B-$module, $F \in \mathcal{L}(\mathcal{E})$ is a homogeneous operator of degree 1, and $\forall a \in A$:

$$
\begin{aligned}
&(\mathrm{I}) \quad \pi(a)(F^2 - 1) \in C(\mathcal{E}),\\
&(\mathrm{II}) \;\; [\pi(a), F] \in C(\mathcal{E})
\end{aligned}
$$

where $C(\mathcal{E})$ is the algebra of compact operators. A triple $(\mathcal{E}, \pi, F)$ will be called degenerate if $\forall a \in A$: $\pi(a)(F^2-1) = 0$, $[\pi(a), F] = 0$. Let $D(A,B)$ be a set of generated triples. An element $E(A, B[0,1])$, where $B[0,1]$ is an algebra of continuous functions in B on the interval $[0,1]$, will be called a homotopy in $E(A,B)$. Let us assign a direct sum in $E(A,B)$:

$$
(\mathcal{E}, \pi, F) \oplus (\mathcal{E}^{'}, \pi^{'}, F^{'}) = (\mathcal{E} \oplus \mathcal{E}^{'}, \pi \oplus \pi^{'}, F \oplus F^{'})\,.
$$

The homotopy classes of $E(A,B)$ together with this sum define the Abelian group $KK(A,B)$. A $*-$homomorphism, $f : A_1 \to A_2$, transfers $(A_2, B)-$modules into $(A_1, B)-$ modules, and

[33]

$$f^* : E(A_2, B) \to E(A_1, B)\,,\ (\mathcal{E}, \pi, F) \mapsto (\mathcal{E}, \pi^\circ f, F)\,,$$

On the other hand a $(*)-$homomorphism $g : B_1 \to B_2$ induces a homomorphism

$$g_* :\ E(A, B_1) \to E(A, B_2)\,,\quad (\mathcal{E}, \pi, f) \mapsto (\mathcal{E} \otimes_g B_2, \pi \otimes 1, F \otimes 1)\,,$$

where

$$\pi \otimes 1 : A \to \mathcal{L}(\mathcal{E} \otimes_g B_2)\,,\qquad (\pi \otimes 1)(a)(e \otimes b) = \pi(a)e \otimes b\,.$$

Theorem 5 *The groups $KK(A, B)$ define a homotopy invariant bifunctor from the category of separable C^*-algebras into the category of Abelian groups. Abelian groups $KK(A, B)$ depend covariantly on the algebras A and B, in addition $KK(\mathbb{C}, B) = K_0(B)$.*

The facts of interest to us are the following relations:

$$KK^*(A := \mathbb{C}, B) = K_*(B), \qquad KK^*(A, B := \mathbb{C}) = K^*(A). \tag{31}$$

Definition 1 *Let $1_A \in KK(A, A)$ ($KK(A, A)$ is a ring with unit) denotes the triple class $(A, \iota_A, 0)$ where $A^{(1)} = A$, $A^{(0)} = 0$, and $\iota_A : A \to \mathcal{K}(A) \subset \mathcal{L}(A)$, $\iota_A(a)b = ab$, $a, b \in A$.*

Let us define also the map

$$\begin{aligned} &\tau_D :\ KK(A, B) \otimes KK(A \otimes D, B \otimes D)\,, \\ &\tau_D\,(\text{class }(\mathcal{E}, \pi, F)) = \text{class }(\mathcal{E} \otimes D, \pi \otimes 1_D, F \otimes 1)\,. \end{aligned}$$

The most important object of our concern is Kasparov's pairing

Theorem 6 *Kasparov's pairing, defined by*

$$KK(A, D) \times KK(D, B) \longrightarrow KK(A, B) \tag{32}$$

and denoted $(x, y) \mapsto x \otimes_D y$, satisfies the following properties:

- *It depends covariantly on the algebra B and contravariantly on the algebra A.*
- *If $f : D \to E$ is a $*-$homomorphism, then $f_*(x) \otimes_E y = x \otimes_D f^*(y)$, $x \in KK(A, D)$, $y \in KK(E, B)$.*
- *Associative property: $(z \otimes_D y) \otimes_E z = x \otimes_D (y \otimes)_E z$, $\forall x \in KK(A, D)$, $y \in KK(D, E)$, $z \in KK(E, B)$.*
- $x \otimes_B 1_B = 1_A \otimes x = x$, $\forall x \in KK(A, B)$.

- $\tau_E(x \otimes_B y) = \tau_E(x) \otimes_{B\otimes E} \tau_E(y)$, $\forall x \in KK(A,B), \forall y \in KK(B,D)$.

Suppose that for two algebras, A and B, there are elements $\alpha \in KK(A \otimes B, \mathbb{C})$, $\beta \in KK(\mathbb{C}, A\otimes B)$, with the property that $\beta \otimes_A \alpha = 1_B \in KK(B,B)$, $\beta \otimes_B \alpha = 1_A \in KK(A,A)$. Then we say we have KK-duality isomorphisms between the K-theory (K-homology) of the algebra A and the K-homology (K-theory) of the algebra B

$$K_*(A) \cong K^*(B),\ K^*(A) \cong K_*(B)\,.$$

In fact the algebras A and B are Poincaré dual [34], but generally speaking these algebras are not KK-equivalent.

K-amenable groups. We now review the concept of K-amenable groups [35]. Let G be a connected Lie group and $\mathcal{K}$ a maximal compact subgroup. We also assume that dim $(G/\mathcal{K})$ is even and $G/\mathcal{K}$ admits a $G-$invariant $\mathrm{Spin}^{\mathbb{C}}$ structure. The $G-$invariant Dirac operator $\mathfrak{D} := \gamma^\mu\partial_\mu$ on $G/\mathcal{K}$ is a first order self-adjoint, elliptic differential operator acting on L^2- sections of the $\mathbb{Z}_2-$graded homogeneous bundle of spinors $\mathcal{S}$. Let us consider a $0^{\rm th}$ order pseudo-differential operator $\mathcal{O} = \mathfrak{D}(1+\mathfrak{D}^2)^{-\frac{1}{2}}$ acting on $H = L^2(G/\mathcal{K}, \mathcal{S})$. $C(G/\mathcal{K})$ acts on H by multiplication of operators. G acts on $C(G/\mathcal{K})$ and on H by left translation, and $\mathcal{O}$ is $G-$invariant. Then, the set $(\mathcal{O}, H, X)$ defines a canonical Dirac element $\alpha_G = KK_G(C(G/\mathcal{K}), \mathbb{C})$.

Theorem 7 *(G. Kasparov [36]) There is a canonical Mishchenko element*

$$\alpha_G \in KK_G(C(G/\mathcal{K}), \mathbb{C})$$

such that the following intersection products occur:

- $\alpha_G \otimes_{\mathbb{C}} \beta_G = 1_{C(G/\mathcal{K})} \in KK_G(C(G/\mathcal{K}), C(G/\mathcal{K}))$,
- $\beta_G \otimes_{C(G/\mathcal{K})} \alpha_G = \gamma_G = KK_G(\mathbb{C}, \mathbb{C})$ *where* γ_G *is an element in* $KK_G(\mathbb{C}, \mathbb{C})$.

For a semisimple Lie group G or for $G = \mathbb{R}^n$, a construction of the Mishchenko element β_G can be found in [35]. We now come to the basic definition:

Definition 2 *A Lie group G is said to be K-amenable if γ_G=1.*

We recall that a group G is amenable if there exists a left invariant positive linear functional on the space $B(G)$ of all continuous bounded functions on G with the norm $\parallel f \parallel = \sup|fx|$. All solvable groups are amenable, while any non-compact semisimple Lie group is non–amenable. However the non-amenable groups $SO_1(n, 1)$ and $SU(n, 1)$ are K-amenable [37, 38].

5.2 Twisted cohomology and K-theory

.

In the presence of a nontrivial background B-field the classification of charges and RR–fields needs a radical modification. It was argued by Witten, [1], that, when H, the curvature

of B, is pure torsion (flat B), the D-brane charges take value in a twisted version of K-theory. Subsequently in [39], this construction was extended to the case $[H] \neq 0$. In this case it was shown that brane charges are classified by certain infinite dimensional algebra bundles of compact operators, introduced by Dixmier and Douady, [40]. More in detail, it was proposed that in the presence of a non-flat B-field D-brane charges in type IIB string be measured by the twisted K-theory that has been described by Rosenberg, [41], and the twisted bundles on the D-brane world-volumes be elements in this twisted K-theory.

First we would like to note the result given in [42, 43]:

$$K_{\#}(C^*(\mathbb{Z},\sigma)) \cong K_{\#}(C^*(\mathbb{Z}^n)) \cong K_{\#}(\mathbb{T}^n)\,, \tag{33}$$

which holds for any group two-cocycle σ on $\mathbb{Z}^n$. This calculation leads to the twisted group of C^*-algebras $C^*(\mathbb{Z},\sigma)$ (noncommutative tori). Such generalization has been given for K-groups of the twisted group of C^*-algebras of uniform lattice in solvable groups [44]. Let Γ be a lattice in a K-amenable Lie group G, then [35]:

$$K_{\#}(C^*(\Gamma,\sigma)) \cong K^{\#+\dim(C/\mathcal{K})}(\Gamma\backslash G/\mathcal{K},\delta(B_\sigma)), \tag{34}$$

where $K^{\#+\dim G}(\Gamma\backslash G,\delta(B_\sigma))$ denotes the twisted K-theory (see [41]) of a continuous trace C^*-algebra B_σ with spectrum $\Gamma\backslash G$, σ is any multiplier on Γ, while $\delta(B_\sigma) \in H^3(\Gamma\backslash G,\mathbb{Z})$ denotes the Dixmier-Douady invariant of B_σ [40]. Let Γ be a lattice in a K-amenable Lie group G. Then the following formulas hold [35]:

$$K_{\#}(C^*(\Gamma,\sigma)) \;\cong\; K^{\#+\dim G}(C_r^*(\Gamma,\sigma))\,, \tag{35}$$
$$K_{\#}(C^*(\Gamma,\sigma)) \;\cong\; K^{\#+\dim G/\mathcal{K}}(\Gamma\backslash G/\mathcal{K},\delta(B_\sigma)). \tag{36}$$

These formulas have been obtained with the help of K-amenability results [37] and the stabilization theorem [44]. When $\Gamma = \Gamma_g$ is a fundamental group of a Riemannian surface $X_\Gamma = \Sigma_g$ of genus $g > 0$ the Dixmier-Douadly class $\delta(B_\sigma)$ is trivial and we get

$$K_0(C^*(\Gamma_g,\sigma)) \cong K^0(\Sigma_g) \cong \mathbb{Z}^2\,, \quad K_1(C^*(\Gamma_g,\sigma)) \cong K^1(\Sigma_g) \cong \mathbb{Z}^{2g}, \tag{37}$$

which hold for any multiplier σ on Γ_σ.

6 Lower K-groups

In this final section we collect a few more basic results on the lower algebraic K-groups of hyperbolic spaces. Let X_Γ be a real compact oriented three dimensional hyperbolic manifold. Its fundamental group Γ comes with maps to $PSL(2,\mathbb{C}) \equiv SL(2,\mathbb{C})/\{\pm Id\}$; therefore in general one gets a class in $H_3(GL(\mathbb{C}))$. The following result holds [45, 46]: For a field $\mathcal{F}$,

$$K_\ell(\mathcal{F}) \cong H_\ell(GL(\ell,\mathcal{F}))/H_\ell(GL(\ell-1,\mathcal{F}). \tag{38}$$

The group $K_3(\mathcal{F})$ is built out of $K_3(\mathcal{F})$ and the Bloch group $\mathfrak{B}(\mathcal{F})$. [6] Since we are looking for $H_3(GL(2,\bullet))$, the homology invariant of a hyperbolic three-manifold should live in the Bloch group $\mathfrak{B}(\bullet)$. The following result confirmed that statement [48]: A real oriented finite-volume hyperbolic three-manifold $X = X_\Gamma$ has the Bloch invariant $\beta(X) \in \mathfrak{B}(\mathbb{C})$. Actually $\beta(X) \in \mathfrak{B}(\mathcal{F})$ for an associated number field $\mathcal{F}(X)$. In fact, under the (normalized) Bloch regulator $\mathfrak{B}(\mathbb{C}) \to \mathbb{C}/\mathbb{Q}$, the invariant $\beta(X)$ goes to $\{(2/\pi)\mathrm{vol}(X) + 4\pi\sqrt{-1}CS(X)\}$. Let us assume that $\mathcal{F}(X)$ can be embedded in $\mathbb{C}$ as an imaginary quadratic extension of a totally real number field; then the Chern-Simons action, $CS(X)$, is rational (conjecturably, $CS(X)$ is irrational if $\mathcal{F}(X) \cap \overline{\mathcal{F}(X)} \subset \mathbb{R}$ [49]).

Finally, taking into account the Thurston classification of all possible three-geometries one can combined the volume and the Chern-Simons invariants as a single complex invariant [50]. By geometry or a geometric structure we mean a pair (X, Γ), that is a manifold X and a group Γ acting transitively on X with compact point stabilizers (we also propose that the interior of every compact three-manifold has a canonical decomposition into pieces which have geometric structure). Then $\Gamma_n\backslash G_n,\ \ n \in (1, ..., 8)$, represents one of the eight geometries, where Γ_n is the (discrete) isometry group of the corresponding geometry. Therefore, Thurston's complex invariant can be presented in the form $\exp\left\{\bigcup_{\ell=1}^{\infty}\left[(2/\pi)\mathrm{Vol}(\Gamma_{n\ell}\backslash G_{n\ell}) + 4\pi\sqrt{-1}CS(\Gamma_{n\ell}\backslash G_{n\ell})\right]\right\}$.

7 Appendix

This appendix is devoted to a collection of basic definitions and results on topological K-theory, following [13]. The K-group of a compact manifold X is the Grothendieck group constructed out classes $[(E, F)]$ of vector bundles E and F over X, with respect to the equivalence relation: $(E, F) \sim (E', F')$ if there exists a vector bundle H on X such that $E + F' + H \cong E' + F + H$. One important elementary property of K-groups is that, if $f, g : X \to Y$ are two homotopic maps, they induce the same map between the corresponding K-groups.

Reduced K-theory. Taking into account that a vector bundle over a point is just a vector space, $K(\mathrm{pt}) = \mathbb{Z}$, we can introduce a reduced K-theory in which the topological space consisting of a single point has trivial cohomology, $\widetilde{K}(\mathrm{pt}) = 0$, and also $\widetilde{K}(X) = 0$ for any contractible space X. Let us consider the collapsing and inclusion maps: $p : X \to \mathrm{pt}\,,\ \iota : \mathrm{pt} \hookrightarrow X$ for a fixed base point of X. These maps induce an epimorphism and a monomorphism of the corresponding K-groups: $p^* : K(\mathrm{pt}) = \mathbb{Z} \to K(X)\,,\ \iota^* : K(X) \to K(\mathrm{pt}) = \mathbb{Z}$. The exact sequences of groups are:

$$0 \to \mathbb{Z} \xrightarrow{p^*} K(X) \to \widetilde{K}(X) \to 0\,, \quad 0 \to \widetilde{K}(X) \to K(X) \xrightarrow{\iota^*} \mathbb{Z}\,.$$

[6] There is an exact sequence due to Bloch and Wigner:

$$0 \to \mathbb{Q}/\mathbb{Z} \to H_3(PSL(2,\mathbb{C});\mathbb{Z}) \to \mathfrak{B}(\mathbb{C}) \to 0$$

The Bloch group $\mathfrak{B}(\mathbb{C})$ is known to be uniquely divisible, so it has canonically the structure of a $\mathbb{Q}-$vector space [47]. The $\mathbb{Q}/\mathbb{Z}$ in the Bloch-Wigner exact sequence is precisely the torsion of $H_3(PSL(2,\mathbb{C});\mathbb{Z})$.

The kernel of the map i^* (or the cokernel of the map p^*) is called the *reduced K-theory group* and is denoted by $\widetilde{K}(X)$, $\widetilde{K}(X) = \ker \iota^* = \operatorname{coker} p^*$. Therefore, there is the fundamental decomposition $K(X) = \mathbb{Z} \oplus \widetilde{K}(X)$. When X is not compact, we can define $K_c(X)$, the K-theory with compact support. It is isomorphic to $\widetilde{K}(X)$.

Higher K-groups and Bott periodicity. The *higher* K-groups are labelled by a positive integer $\ell = \mathbb{Z}_+$ and can be defined according to $K^{-\ell}(X) = K(\Sigma^\ell X)$, where $\Sigma^\ell X \equiv \mathbb{S}^\ell \wedge X$ is the $\ell-$th reduced suspension of the topological space X. In addition, $X \wedge Y = X \times Y/(X \vee Y)$ is the smash product of X and Y, where $X \vee Y$ is the reduced join (the disjoint union with a base point in each space identified). The higher K-groups (with compact support K_c) can also be defined through the suspension isomorphism: $K^{-\ell}(X) = K(X \times \mathbb{R}^\ell)$.

The *Bott periodicity theorem*, states that the complex K-theory functor $K^{-\ell}$ is periodic with period two:

$$K^{-\ell}(X) = K^{-\ell-2}(X). \tag{39}$$

The same statement holds for the reduced functor $\widetilde{K}^{-\ell}$. This periodicity property is at variance with the conventional cohomology theories. Note however, that the higher reduced and unreduced K-groups differ according to $K^{-\ell}(X) = \widetilde{K}^{-\ell}(X) \oplus K^{-\ell}(\mathrm{pt})$. It follows that for the decomposition $X = X_1 \amalg X_2 \amalg \cdots \amalg X_\ell$ of X into a disjoint union of open subspaces, the inclusions of the X_j into X induce a decomposition of K-groups as $K^{-\ell}(X) = K^{-\ell}(X_1) \oplus K^{-\ell}(X_2) \oplus \cdots \oplus K^{-\ell}(X_\ell)$. This statement is not true for the reduced K-functor.

Multiplicative structures. Since $K(X)$ and $\widetilde{K}(X)$ are rings (as in any cohomology theory), the multiplication is induced by the tensor product $E \otimes F$ of vector bundles over $X \times X$ (see for detail [5, 13]):

$$K(X) \otimes_{\mathbb{Z}} K(X) \to K(X). \tag{40}$$

There is a homomorphism called the external tensor product or *cup product*,

$$K(X) \otimes_{\mathbb{Z}} K(Y) \to K(X \times Y)\,. \tag{41}$$

It is defined as follows. Consider the canonical projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$, which induce homomorphisms between K-groups: $\pi_X^* : K(X) \to K(X \times Y)\,, \pi_Y^* : K(Y) \to K(X \times Y)$. The cup product of $([E],[F]) \in K(X) \otimes_{\mathbb{Z}} K(Y)$ is the class $[E] \otimes [F]$ in $K(X \times Y)$, where $[E] \otimes [F] \equiv \pi_X^*([E]) \otimes \pi_Y^*([F])$. If we consider the canonical injective inclusion and surjective projection maps, $X \vee Y \hookrightarrow X \times Y \to X \wedge Y$, then the contravariant functor $\widetilde{K}^{-\ell}$ induces a split short exact sequence of K-groups:

$$0 \longrightarrow \widetilde{K}^{-n}(X \wedge Y) \longrightarrow \widetilde{K}^{-n}(X \times Y) \longrightarrow \widetilde{K}^{-n}(X \vee Y) \longrightarrow 0\,.$$

Then a useful formula for computing the K-groups of Cartesian products [13] follows:

$$\begin{aligned} \widetilde{K}^{-\ell}(X \times Y) &= \widetilde{K}^{-\ell}(X \wedge Y) \oplus \widetilde{K}^{-\ell}(X \vee Y) \\ &= \widetilde{K}^{-\ell}(X \wedge Y) \oplus \widetilde{K}^{-\ell}(X) \oplus \widetilde{K}^{-\ell}(Y). \end{aligned} \tag{42}$$

As an example, let $Y = \mathbb{S}^1$. $K^{-1}(X)$ can be identified with the set of K-theory classes in $\widetilde{K}(X \times \mathbb{S}^1)$ which vanish when restricted to $X \times \mathrm{pt}$. Since $\widetilde{K}(\mathbb{S}^1) = 0$, $\widetilde{K}(\mathbb{S}^1 \wedge X) = K^{-1}(X)$, we get

$$\widetilde{K}(X \times \mathbb{S}^1) = \widetilde{K}(X \wedge \mathbb{S}^1) \oplus \widetilde{K}(X) \oplus \widetilde{K}(\mathbb{S}^1) = K^{-1}(X) \oplus \widetilde{K}(X)\,, \tag{43}$$

$$\begin{aligned} K^{-1}(X \times \mathbb{S}^1) &= K^{-1}(X \wedge \mathbb{S}^1) \oplus K^{-1}(X) \oplus K^{-1}(\mathbb{S}^1) \\ &= \widetilde{K}(X) \oplus K^{-1}(X) \oplus \mathbb{Z}\,. \end{aligned} \tag{44}$$

Using the action of cup product on reduced K-theory we can also obtain the following formula: $(\widetilde{K}(X) \otimes_{\mathbb{Z}} \widetilde{K}(Y)) \oplus \Xi \to \widetilde{K}(X \wedge Y) \oplus \Xi$, here $\Xi = \widetilde{K}(X) \oplus \widetilde{K}(Y) \oplus \mathbb{Z}$. The group Ξ appears on both sides of the previous formula, and we can eliminate it (by an appropriate restriction) and get the homomorphism $\widetilde{K}(X) \otimes_{\mathbb{Z}} \widetilde{K}(Y) \to \widetilde{K}(X \wedge Y)$. If $K(X)$ or $K(Y)$ is a free abelian group, this mapping and the cup product are isomorphisms. Define $K^{\#}(X)$ to be the $\mathbb{Z}_2-$graded ring $K^{\#}(X) = K(X) \oplus K^{-1}(X)$. If $K^{\#}(X)$ or $K^{\#}(Y)$ is freely generated, the K-theory analog of the cohomological Künneth theorem holds: $K^{\#}(X \times Y) = K^{\#}(X) \otimes_{\mathbb{Z}} K^{\#}(Y)$. In general, however, there are correction terms on the right-hand side of this formula, which relate to the torsion subgroups of the K-groups [51]. The case of torsion-free subgroups also leads to

$$K(X \times Y) = (K(X) \otimes_{\mathbb{Z}} K(Y)) \oplus (K^{-1}(X) \otimes_{\mathbb{Z}} K^{-1}(Y))\,, \tag{45}$$

$$K^{-1}(X \times Y) = (K(X) \otimes_{\mathbb{Z}} K^{-1}(Y)) \oplus (K^{-1}(X) \otimes_{\mathbb{Z}} K(Y))\,. \tag{46}$$

Relative K-groups. Let us suppose that Y is a closed submanifold of X, then the K-group $K(X, Y)$ can be defined, whose classes are identified with pairs of bundles over X/Y. If $Y \neq \emptyset$, then the topological coset X/Y is defined to be the space X with Y shrunk to a point; if Y is empty then we can identify X/Y with the one-point compactification X^+ of X. We can construct a one-to-one correspondence between vector bundles over the quotient space X/Y and vector bundles E over X whose restriction to Y is a trivial bundle. Then the *relative K-group* is defined as $K(X, Y) \equiv \widetilde{K}(X/Y)$. $K(X, Y)$ is a contravariant functor of the pair (X, Y) and, since $K(X) = \widetilde{K}(X) \oplus K(pt) = \widetilde{K}(X^+)$, we have $K(X, \emptyset) = K(X)$. For the relative K-groups the Bott periodicity holds: $K^{-\ell}(X, Y) = K^{-\ell-2}(X, Y)$. One of the most important properties of K-groups is that they satisfy the Barratt-Puppe exact sequence:

$$\ldots \longrightarrow K^{-\ell-1}(X) \longrightarrow K^{-\ell-1}(Y) \xrightarrow{\partial^*} K^{-\ell}(X, Y) \longrightarrow K^{-\ell}(X) \longrightarrow K^{-\ell}(Y) \xrightarrow{\partial^*} \ldots, \tag{47}$$

where ∂ is the boundary homomorphism. The sequence (47) connects the K-groups of X and $Y \subset X$ and makes K-theory similar to a cohomology theory. Let $\iota : Y \to X$ and $j : (X, \emptyset) \to (X, Y)$ be inclusions. Then, $K(X, Y) \xrightarrow{j^*} K(X) \xrightarrow{\iota^*} K(Y)$. If we suppose now that Y is a retract of X, it means that the inclusion map $\iota : Y \to X$ admits a left inverse, then the last sequence splits, giving

$$K^{-\ell}(X) = K^{-\ell}(X, Y) \oplus K^{-\ell}(Y)\,. \tag{48}$$

Equivariant K-theory. The natural K-theoretic methods for classification of D-branes in orbifolds (and orientifolds) of the Type II and Type I theories is equivariant K-theory. Let X

be a smooth manifold and G a group acting on X (G is either a finite group or a compact Lie group). Thus, X is a *G-manifold* and we can write the G-action $G \times X \to X$ as $(g,x) \mapsto g \cdot x$. Suppose that a *G-map* $f : X \to Y$ between two G-manifolds is a smooth map which commutes with the action of G on X and Y: $f(g \cdot x) = g \cdot f(x)$. Then we say that f is G-equivariant. A principal fiber bundle $E \to X$ is a *G-bundle* when E is a G-manifold and the canonical fiber projection π is a G-map: $\pi(g \cdot v) = g \cdot \pi(v),\ \forall v \in E,\ g \in G$. A *G-isomorphism* is a map $E_G \to F_G$ between G-bundles over X, which is both a bundle isomorphism and a G-map. It defines the category [7] of G-equivariant bundles over the G-space X. The Grothendieck group of this theory is called G-equivariant and denoted $K_G(X)$. $K_G(X)$ consists of pairs of bundles (E,F) with G-action, modulo the equivalence relation $(E,F) \sim (E \oplus H, F \oplus H)$ for any G-bundle H over X. In this way D-brane configurations on X/G are understood as G-invariant configurations of D-branes on X, [52], in other words, the orbifold spacetime is regarded as a G-space. For type IIB superstrings on an orbifold X/G, the D-brane charge takes values in $K_G(X)$. For type IIA one has $K_G^{-1}(X)$ and for type I we have $KO_G(X)$ (see below). We get: $K_G^{-1}(X) \equiv K_G(\Sigma X) = K_G(\mathbb{S}^1 \wedge X)$ with G acting trivially on $\mathbb{S}^1$. For trivial action of G on X one gets, [13], $K_G(X) = K(X) \otimes \mathrm{R}(G)$, where $K(X)$ is the ordinary K-group of X and $\mathrm{R}(G)$ is the representation ring og G. For any compact G-space X, the collapsing map $X \to$ pt gives rise to an $\mathrm{R}(G)-$module structure on $K_G^{\#}(X)$, such that $\mathrm{R}(G)$ is the coefficient ring in the equivariant K-theory (instead of $\mathbb{Z}$ as in the ordinary case). K_G is functorial with respect to group homomorphisms. Since $K_G(X)$ is a generalization of the two important classification groups $K(X)$ and $\mathrm{R}(G)$, the equivariant K-theory unifies K-theory and group representation theory. For the trivial space $X = \mathrm{pt}$ one has $K_G(X) = \mathrm{R}(G)$ while the trivial group $G = \mathrm{Id}$ leads to $K_G(X) = K(X)$. If H is a closed subgroup of G, then for any H-space X, the inclusion $\iota : H \hookrightarrow G$ induces an isomorphism $\iota^* : K_G(G \times_H X) \xrightarrow{\approx} K_H(X)$.

Let the group G act freely on X. For instance, let $G = \Gamma$ be a co-compact group acting on real hyperbolic spaces $X = \mathbb{H}^N$ without fixed points, giving rise to compact spaces X_Γ. X_Γ is a topological space and its $\Gamma-$equivariant K-theory is just $K_\Gamma(\mathbb{H}^N) = K(\Gamma\backslash\mathbb{H}^N)$.

In general (X/G is not a topological space) there is a useful exact sequence for computing equivariant K-theory:

$$
\begin{aligned}
K_G^{-1}(X,Y) &\longrightarrow K_G^{-1}(X) \longrightarrow K_G^{-1}(Y) \xrightarrow{\partial^*} K_G(X,Y) \\
K_G^{-1}(X,Y) &\xleftarrow{\partial^*} K_G(Y) \longleftarrow K_G(Y) \longrightarrow K_G(X) \longleftarrow K_G(X,Y)
\end{aligned}
\tag{49}
$$

Here Y is a closed G-subspace of a locally compact G-space X, and the relative K-theory is defined by $K_G^{-\ell}(X,Y) = \widetilde{K}_G^{-\ell}(X/Y)$.

Real K-groups. Let us consider pairs of bundles (E,F) with $(E \oplus H, F \oplus H)$ for any $SO(K)$ bundle H. Pairs (E,F) with this equivalence relation define the real $K-$group $KO(X)$ of the

[7] A *category* $\mathcal{C}$ consists the following data: 1) A class Ob $\mathcal{C}$ of objects $A, B, C, ...$; 2) A family of disjoint sets of morphisms $\mathrm{Hom}(A,B)$, one for each ordered pair A, B of objects; 3) A family of maps $\mathrm{Hom}(A,B) \times \mathrm{Hom}(B,C) \to \mathrm{Hom}(A,C)$, one for each ordered triplet A, B, C of objects. These data obey the axioms: a) If $f : A \to B$, $g : B \to C$, $h : C \to D$, then composition of morphisms is associative, that is, $h(gf) = (hg)f$; b) To each object B there exists a morphism $1_B : B \to B$ such that $1_B f = f$, $g 1_B = g$ for $f : A \to B$ and $g : B \to C$.

spacetime X. Alternatively we can introduce the reduced real K-theory group $\widetilde{KO}(X)$. It follows from the bound state construction that D-brane configurations of Type I superstring theory are classified by $KO(X)$ with compact support [1]. The torsion KO-groups modify various product relations that have been considered above. Using, for example, (42) we get the analog of (43):

$$\widetilde{KO}(X \times \mathbb{S}^1) = \widetilde{KO}^{-1}(X) \oplus \widetilde{KO}(X) \oplus \mathbb{Z}_2 \,. \tag{50}$$

Remember that the Grothendieck group of all virtual bundles with involutions on X is called the real K-group $KR(X)$. In the standard way one can define higher groups $KR^{-m}(X)$ by $\widetilde{KR}^{-\ell}(X) = \widetilde{KR}(X \wedge \mathbb{S}^\ell)$, with the involution τ on X, i.e. a homeomorphism $\tau : X \to X$, $\tau^2 = Id_X$, extended to $X \wedge \mathbb{S}^\ell$ by a trivial action on $\mathbb{S}^\ell$. Let $\mathbb{Z}^{p,q}$ be the $(p+q)-$dimensional real space in which an involution acts as a reflection of the last q coordinates: for given $(x, y) \in \mathbb{R}^p \times \mathbb{R}^q$ we get $\tau : (x, y) \mapsto (x, -y)$. Suppose $\mathbb{S}^{p,q}$ is the unit sphere of dimension $p + q - 1$ in $\mathbb{R}^{p,q}$ with respect to the flat Euclidean metric on $\mathbb{R}^p \times \mathbb{R}^q$. One can define a two-parameter set of higher degree KR-groups [13]: $\widetilde{KR}^{p,q}(X) = \widetilde{KR}(X \wedge \mathbb{R}^{p,q}_+)$ and $KR^{p,q}(X) = KR(X \times \mathbb{R}^{p,q})$. Then we have $KR^{-\ell}(X) = KR^{\ell,0}(X)$, and Bott periodicity in KR-theory has the form

$$KR^{p,q}(X) = KR^{p+1,q+1}(X)\,, \quad KR^{-\ell}(X) = KR^{-\ell-8}(X). \tag{51}$$

¿From this formula we get: $KR^{p,q}(X) = KR^{q-p}(X)$, and $KR^{p,q}(X)$ only depends on the difference $p - q$. Also, $KR^{p,q}(X)$ depends only on this difference modulo 8. In fact, we can define negative-dimensional spheres as those with antipodal involutions in KR-theory, with $\mathbb{S}^{\ell,0}$ being identified as $\mathbb{S}^{\ell-1}$ and $\mathbb{S}^{0,\ell}$ as $\mathbb{S}^{-\ell-1}$. Identifying $\mathbb{R}^{1,1} = \mathbb{C}$ with the involution τ acting as the complex conjugation, one gets the $(1, 1)$ periodicity theorem in the following form $KR(X) = KR(X \times \mathbb{C})$, which holds for any locally compact space X. For trivial $\tau-$action on X we have

$$KR^{-\ell}(X \times \mathbb{S}^{0,1}) = K^{-\ell}(X)\,, \quad KR^{-\ell}(X) = KO^{-\ell}(X)\,. \tag{52}$$

Most of the properties discussed above for K-groups have obvious counterparts in the real case. For example, by repeating the calculations which give (44) we can obtain, for a trivial action of τ on X, the product formula [13]:

$$\begin{aligned} \widetilde{KR}^{-1}(X \times \mathbb{S}^{1,1}) &= \widetilde{KR}^{-1}(X \wedge \mathbb{S}^{1,1}) \oplus \widetilde{KR}^{-1}(X) \oplus \widetilde{KR}^{-1}(\mathbb{S}^{1,1}) \\ &= \widetilde{KR}^{1,1}(X) \oplus \widetilde{KO}^{-1}(X) \oplus \mathbb{Z} = \widetilde{KO}(X) \oplus \widetilde{KO}^{-1}(X) \oplus \mathbb{Z}. \end{aligned} \tag{53}$$

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